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Equilibriumlike fluctuations in some boundary-driven open diffusive systems

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There exist some boundary-driven open systems with diffusive dynamics whose particle current fluctuations exhibit universal features that belong to the Edwards-Wilkinson universality class. We achieve this result by establishing a mapping, for the system fluctuations, to an equivalent open yet equilibrium-diffusive system. We discuss the possibility of observing dynamic phase transitions using the particle current as a control parameter.

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I. INTRODUCTION

In this work we consider systems of interacting particles undergoing diffusive dynamics, such as the symmetric simple exclusion process (SSEP) [1–4]. Such systems that can be described by the theory of fluctuating hydrodynamics, a coarse-grained description in terms of continuous degrees of freedom living in a continuous space, have already been the subject of intense investigation. For instance, Bertini *et al.* [5–9] relied on fluctuating hydrodynamics to provide a quantitative analysis of large deviation properties of diffusive systems taken out of equilibrium by means of a boundary drive. Among the various large deviation properties investigated so far, those of the particle current play a special role. Indeed, current fluctuations have been known for a long time to be a central quantity since the work of Einstein [10,11] which established that in equilibrium, the current variance is proportional to the diffusion constant; current fluctuations characterize the likeliness of and quantify the system excursions out of equilibrium. In the last decade, generic properties of these large deviation functions were discovered such as the fluctuation theorem which determines how the large deviation function of the current is changed under time reversal [12–20]. Parallel approaches [21–26] which have been employed have revealed the possibility of new types of phase transitions where the particle current plays the role of a control parameter. When looking at the properties of the trajectories in configuration space with a given particle current, the system is uniform for values of the current not too far from its average. However, for large enough values of the current, the system may accommodate the particle flux by breaking translation invariance. This current controlled transition is signalled by a singularity of the current large deviation function [8,9,25]. It must also be mentioned that some results regarding the particle current statistics originate from exact solutions; this is the case for the totally asymmetric exclusion process [27]—a version of the SSEP where the motion of the particles is strongly biased—or for the SSEP [28] with periodic boundary conditions.

Fluctuating hydrodynamics not only encompasses interacting particle systems but also applies to models involving at the microscopic level already continuous degrees of freedom, such as the Kipnis-Marchioro-Presutti [29,30] (KMP) model of interacting harmonic oscillators that has served as a test bench to investigate the statistical properties of heat conduction. Both the SSEP and the KMP models will be the main focus of our efforts in the sequel, though we shall strive to keep our discussion general when possible.

Our central motivation is to investigate the role of finite-size effects in one-dimensional open driven diffusive systems. We have been inspired by the results of Appert-Rolland *et al.* [28] and by those of Derrida and Lebowitz [27]. In the former universal properties were seen to emerge in the statistics of the particle current for an equilibrium-diffusive system with periodic boundary conditions, with the possibility, for certain classes of diffusive systems, to exhibit a current-driven dynamic phase transition. In the latter, where mutually excluding particles are subjected to a bulk electric field that drives the system far from equilibrium, universal features belonging to a different universality class have also been observed. Besides, Bodineau and Derrida [25,26] showed that for a weakly bulk and boundary-driven SSEP, a dynamic phase transition takes place.

In the present work our interest goes to open systems, maintained out of equilibrium by putting them in contact with particle reservoirs at unequal chemical potentials, but the bulk dynamics itself remains reversible. Thus the non-equilibrium nature of our systems does not arise from an external bulk field but only from a boundary drive. We ask the following questions. (i) Do universal features in the particle current appear in an open system? If so, do they depend on the system being possibly driven out of equilibrium by a chemical-potential gradient? (ii) Is the existence of a particle current capable of inducing a dynamic phase transition?

Before entering the technicalities of our work, we would like to phrase the answers we have come up to issues (i) and (ii). To question (i) we have the partial answer that at least for a class of systems—to which the SSEP and the KMP model belong—the current distribution does indeed display universal features. The latter do not depend on the system being in or out of equilibrium and, quite remarkably, they are the same for an open system as for a closed system [28]. They belong to the Edwards-Wilkinson universality class [31]. To question (ii) the answer is not straightforward; we have found at least a family of physical systems (among which the SSEP) in which the current large deviation function displays some singularity, indicating the existence of a dynamic phase transition depending on the scaling of the current one forces through.

We shall begin in Sec. II by recalling what is known on the statistics of the current in an open boundary-driven diffusive system. Section III is devoted to the careful analysis of finite-size effects that lead to establishing that in some cases fluctuations exhibit universal features. This section is supplemented by Appendix B that describes the cases of the SSEP and the KMP model in detail. Our conclusions and yet open problems are gathered in Sec. IV.

II. CURRENT LARGE DEVIATIONS IN DIFFUSIVE IN BOUNDARY-DRIVEN OPEN SYSTEMS

We consider a one-dimensional lattice with L sites, whose state at time t is characterized by the local numbers $n_j(t)$'s (which may be discrete or continuous variables), $j=1,\ldots,L$. Our starting point is the assumption that in the large t and L limit, with t/L^2 fixed, there exists a Langevin equation for a density field $\rho(x,\tau)=n_j(t')$, with $x=j/L$ and $\tau=t'/L^2$ defined over $x\in[0,1]$ and $\tau\in[0,t/L^2]$ which evolves according to

$$\partial_\tau\rho = \partial_x[D(\rho(x,\tau))\partial_x\rho(x,\tau) - \xi(x,\tau)], \tag{1}$$

where the Gaussian white noise ξ has correlations $\langle\xi(x,\tau)\xi(x',\tau')\rangle=\frac{\sigma(\rho(x,\tau))}{L}\delta(x-x')\delta(\tau-\tau')$ that decay to zero as the inverse system size. The phenomenological coefficients $D(\rho)$ (the diffusion constant) and $\sigma(\rho)$ depend on some of the details of the underlying microscopic dynamics. The system size L and the observation time t are large with respect to microscopic space and time scales. The system is in contact at both ends with reservoirs that fix the value of ρ at all times to be ρ_0 at $x=0$ and ρ_1 at $x=1$.

Our interest lies in the statistics of the total particle current $Q(t)$ accumulated up until time t and its large deviation properties, which, in terms of field ρ, is formally expressed as

$$Q(t) = L^2\int_0^1 dx\int_0^{t/L^2} d\tau[-D(\rho(x,\tau))\partial_x\rho(x,\tau) + \xi(x,\tau)]. \tag{2}$$

Our purpose is to determine

$$\pi(j) = \lim_{t\to\infty}\frac{\ln \text{Prob}\{Q(t)=jt\}}{t} \tag{3}$$

or, equivalently, its Legendre transform $\psi(s)=\max_j\{\pi(j)-sj\}$ that can be obtained from the generating function of Q as follows:

$$\psi(s) = \lim_{t\to\infty}\frac{\ln\langle e^{-sQ(t)}\rangle}{t}. \tag{4}$$

Using the Janssen–De Dominicis formalism [32], we see that the generating function $\langle e^{-sQ(t)}\rangle$ can be cast in the form of a path integral over two fields,

$$\langle e^{-sQ(t)}\rangle = \int \mathcal{D}\bar{\rho}\mathcal{D}\rho e^{-LS[\bar{\rho},\rho]}, \tag{5}$$

where the action S is given by

$$S[\bar{\rho},\rho] = \int_0^1 dx\int_0^{t/L^2} d\tau\Big[\bar{\rho}\partial_\tau\rho + D(\rho)\partial_x\bar{\rho}\partial_x\rho - \frac{\sigma}{2}(\partial_x\bar{\rho}-\lambda)^2 - \lambda D\partial_x\rho\Big], \tag{6}$$

where $\lambda=sL$ and the path integral runs over functions verifying the boundary conditions $\rho(0,\tau)=\rho_0$, $\rho(1,\tau)=\rho_1$, and $\bar{\rho}(0,\tau)=\bar{\rho}(1,\tau)=0$. It is very clear from the expression of the noise in Eq. (1) or from path integral (5) that a semiclassical-like expansion is valid in the weak-noise limit, which, translated in our language, is synonymous for a large system-size expansion. In short, path integral (5) is dominated by the saddle point of S. The reader is referred to Kurchan's lectures [33] or to [34] for a pedagogical account exploiting this language. We first change the response field into $\tilde{\rho}(x,\tau)=\bar{\rho}(x,\tau)-\lambda x$. We write the saddle-point equations and we assume that the saddle is reached for time-independent profiles $\tilde{\rho}_c(x),\rho_c(x)$. Sufficient conditions under which this is so were discussed by Bertini *et al.* [9]. Assuming this is indeed the case, the saddle-point equations $\frac{\delta S}{\delta\rho}=0$ and $\frac{\delta S}{\delta\bar{\rho}}=0$ read

$$\partial_\tau\rho = \partial_x(D\partial_x\rho) - \partial_x(\sigma\partial_x\tilde{\rho}), \quad -\partial_\tau\tilde{\rho} = \partial_x(D\partial_x\tilde{\rho}) + \frac{\sigma'}{2}(\partial_x\tilde{\rho})^2, \tag{7}$$

which, assuming a stationary solution, lead to

$$D^2(\rho_c)(\partial_x\rho_c)^2 = K_1^2 + K_2\sigma(\rho_c), \quad \partial_x\tilde{\rho}_c = \frac{D(\rho_c)\partial_x\rho_c + K_1}{\sigma(\rho_c)}, \tag{8}$$

where K_1 and K_2 are λ-dependent constants. With these equations, one may verify that the action evaluated at the saddle reads $S[\tilde{\rho}_c,\rho_c]=(t/L^2)K_2/2$. We shall denote by $\mu(\lambda)=-K_2/2$ (our definition of μ differs from that of [24,35] by a factor of $1/L$). With these notations, for the large deviation function introduced in Eq. (4), we thus have $\psi(s)=\frac{\mu(sL)}{L}$. In practice, to explicitly determine $\mu(\lambda)$, one must solve the differential equations [Eqs. (8)] and fix the constants K_1 and K_2 by means of the appropriate boundary conditions. A few comments are in order; these results are not new and they were first derived by Bodineau and Derrida [24]. We propose as an illustration the explicit expression for $\mu(\lambda)$ when the diffusion constant D is independent of the local density (we take $D=1$) and when the noise strength $\sigma(\rho)$ is a simple quadratic function. For $\sigma(\rho)=c_2\rho^2+c_1\rho$ we find that (see Appendix A)

$$\mu(\lambda) = \begin{cases} -\dfrac{2}{c_2}(\text{arcsinh}\sqrt{\omega})^2 & \text{for } \omega>0 \\ +\dfrac{2}{c_2}(\arcsin\sqrt{-\omega})^2 & \text{for } \omega<0, \end{cases} \tag{9}$$

where $\omega(\lambda,\rho_0,\rho_1)$ is the auxiliary variable given by

$$\omega(\lambda,\rho_0,\rho_1) = \frac{c_2}{c_1^2}(1-e^{c_1\lambda/2})[c_1(\rho_1 - e^{-c_1\lambda/2}\rho_0) - c_2(e^{-c_1\lambda/2}-1)\rho_0\rho_1]. \quad (10)$$

For the SSEP, $\sigma(\rho)=2\rho(1-\rho)$ and one recovers the known [35,36] result (the notation $z=e^{-\lambda}$ is used in formula (2.14) of [35]), namely,

$$\omega(\lambda,\rho_0,\rho_1) = (1-e^{\lambda})[e^{-\lambda}\rho_0 - \rho_1 - (e^{-\lambda}-1)\rho_0\rho_1]. \quad (11)$$

Another solvable model is the KMP chain of coupled harmonic oscillators, for which $D=1$ and $\sigma(\rho)=4\rho^2$ (for KMP, ρ stands for the local potential-energy field), for which we also have Eq. (9) but where the variable ω is now given by

$$\omega(\lambda,\rho_0,\rho_1) = \lambda[2(\rho_0-\rho_1) - 4\lambda\rho_0\rho_1]. \quad (12)$$

There exists a set of numerical simulations by Hurtado and Garrido [37] which agree with the formulas [Eqs. (9) and (12)] for the KMP model for values of λ not too close from the boundaries of the domain of definition. Cases $c_2>0$ and $c_2<0$ are qualitatively different. In the latter, $\mu(\lambda)$ is defined over the whole real axis and is unbounded from above, while in the former $\mu(\lambda)$ is defined over a finite interval of λ whose ends correspond to infinite currents produced by the build up of infinite densities. For example, with $c_2=4$ and $c_1=0$, that is, for the KMP model, $-\frac{1}{2\rho_1}<\lambda<\frac{1}{2\rho_0}$.

Finally, it is important to realize that $\mu(\lambda)$ is the leading-order term in a large system-size series expansion. The origin of finite-size corrections is twofold. Of course there will be finite-size corrections arising from integrating out the modes describing fluctuations around the optimal profile $\{\tilde{\rho}_c,\rho_c\}$. However, fluctuating hydrodynamics, by definition, is unable to capture the details of the microscopic systems it describes. It must therefore be expected that model-dependent finite-size corrections will also emerge. We now proceed with determining the finite-size contribution of fluctuations around the saddle of action (6) within the framework of fluctuating hydrodynamics (that is, temporarily omitting contributions arising from the underlying discreteness of the lattice).

III. FLUCTUATIONS AND UNIVERSAL BEHAVIOR

A. Evaluating a determinant

As in any saddle-point calculation, we obtain finite-size corrections to the leading-order result $\langle e^{-sQ}\rangle \simeq e^{\mu(sL)t}$ by introducing, in path integral (5) the fluctuations around the optimal profile $\tilde{\rho}_c$ and ρ_c: $\phi(x,\tau)=\rho(x,\tau)-\rho_c(x)$ and $\bar{\phi}(x,\tau) =\tilde{\rho}(x,\tau)-\tilde{\rho}_c(x)$. Then we expand action (6) to quadratic order in ϕ and $\bar{\phi}$,

$$S = -\frac{\mu(\lambda)t}{L^2} + \int dx d\tau\bigg(\bar{\phi}\partial_\tau\phi + D\partial_x\bar{\phi}\partial_x\phi + D'\partial_x\tilde{\rho}_c\phi\partial_x\phi + D'\partial_x\rho_c\partial_x\bar{\phi}\phi + \frac{D''}{2}\partial_x\tilde{\rho}\partial_x\rho\phi^2 - \frac{\sigma}{2}(\partial_x\bar{\phi})^2 - \sigma'\partial_x\tilde{\rho}_c\phi\partial_x\bar{\phi} - \frac{\sigma''}{4}(\partial_x\tilde{\rho}_c)^2\phi^2\bigg), \quad (13)$$

where D, σ, and their derivatives with respect to the density are evaluated at $\rho_c(x)$. The goal is to integrate out the quadratic action (13) with respect to the fields $\bar{\phi}$ and ϕ. This is the procedure that was followed in [28] and that we carry out here as well. However, unlike the case of periodic boundary conditions dealt with in [28], in the present case, the quadratic action is not readily diagonalizable for its coefficients are space dependent. It so happens that for one particular family of models, those for which $D(\rho)$ is constant and $\sigma(\rho)$ is quadratic in ρ, this can actually be achieved. This remains a nontrivial task, given that the quadratic form to diagonalize in Eq. (13) still possesses space-dependent coefficients. We have not been able deal with arbitrary D and σ.

B. Constant D and quadratic σ

We specialize action (13) to a constant D and a quadratic σ. After performing the change in fields,

$$\phi = (\partial_x\tilde{\rho}_c)^{-1}\psi + \partial_x\rho_c\bar{\psi}, \quad \bar{\phi} = \partial_x\tilde{\rho}_c\bar{\psi}, \quad (14)$$

we note that Eq. (13), after tedious rearrangements, becomes

$$S = -\frac{\mu(\lambda)t}{L^2} + \int dx d\tau\bigg(\bar{\psi}\partial_\tau\psi + D\partial_x\bar{\psi}\partial_x\psi - \mu(\lambda)(\partial_x\bar{\psi})^2 - \frac{\sigma''}{4}\psi^2\bigg). \quad (15)$$

It is remarkable that Eq. (15) is now a quadratic form that can be diagonalized with standard stationary waves $\{\sin qx\}_q$ with Fourier modes indexed by $q=\pi n$, where $n\in\mathbb{N}^*$. By comparison to Eq. (13), we can interpret Eq. (15) as being the action corresponding to an *equilibrium* open system whose current fluctuations we study as a function of the conjugate variable $\frac{\mu(sL)}{L}$. Performing the change of variables (14) has allowed us to map the fluctuations onto those of an open system in contact with two reservoirs at equal densities.

After integrating out the ψ and $\bar{\psi}$ fields, one arrives at

$$\psi_{\rm FH}(s) = \frac{1}{L}\mu(sL) + \frac{D}{8L^2}\mathcal{F}\bigg(\frac{\sigma''}{2D^2}\mu(sL)\bigg), \quad (16)$$

where the FH index stands for "fluctuating hydrodynamics" where function $\mathcal{F}$ has the expression

$$\mathcal{F}(u) = -4\sum_{q=n\pi,n\geq 1}(q\sqrt{q^2-2u} - q^2 + u). \quad (17)$$

Equation (16) is the first new result of this work. It indicates that for systems whose fluctuating hydrodynamics description relies on a constant diffusion constant D and a quadratic noise strength $\sigma(\rho)$, current fluctuations involve a universal scaling function $\mathcal{F}$. It is remarkable that exactly the same function $\mathcal{F}$ has appeared in the study of current fluctuations in closed systems *in equilibrium*, with a different scaling variable though. As can be seen from its explicit expression (17), the scaling function $\mathcal{F}$ has a singularity when its argument approaches $\pi^2/2$ from the right real axis. In [28,38] this was interpreted as the presence of a first-order dynamic phase transition for systems with periodic boundary conditions. In the present case, this also opens up the possibility of a similar phase transition on the condition that there exists a regime of λ for which

$$\frac{\sigma''}{2D^2}\mu(\lambda) > \frac{\pi^2}{2}. \tag{18}$$

Before we discuss whether a phase transition can indeed occur, we must address another pending issue.

C. Microscopic contribution

The expression for $\psi_{\rm FH}(\lambda/L)=\frac{1}{L}\mu(\lambda)+\frac{D}{8L^2}\mathcal{F}[\frac{\sigma''}{2D^2}\mu(\lambda)]$ obtained from fluctuating hydrodynamics ignores the possibility that finite-size corrections of the same $\mathcal{O}(L^{-2})$ order as the universal corrections will appear when one relies on the original model defined on a lattice. In Appendix B, which is based on methods developed by Tailleur *et al.* [39], we are able to evaluate the contribution of lattice effects for two specific models. We show that for the SSEP and for the KMP model they do introduce $\mathcal{O}(L^{-2})$ terms that add up to the universal contribution found from fluctuating hydrodynamics. Let us look into those microscopic details more precisely, first for the SSEP, then for the KMP model.

The open and driven SSEP consist of particles hopping to either of their nearest-neighbor sites on a lattice of L sites, in contact with particle reservoirs connected to sites 1 and L. Particles are injected into site 1 (resp. L) with a rate α (resp. δ) and are removed from site 1 (resp. L.) with rate γ (resp. β). These reservoirs impose densities $\rho_0=\frac{\alpha}{\alpha+\gamma}$ and $\rho_1=\frac{\delta}{\beta+\delta}$ at sites 1 and L. While in the fluctuating hydrodynamic formulation the reservoirs enter current statistics through ρ_0 and ρ_1 only, when one wishes to capture phenomena beyond leading order, lattice effects, and microscopic details start playing a role. For the SSEP, as presented in Appendix B, introducing the auxiliary constants $a=\frac{1}{\alpha+\gamma}$ and $b=\frac{1}{\beta+\delta}$, we find that

$$\begin{aligned}\psi(s) &= \psi_{\rm FH}(s) - \frac{a+b-1}{L^2}\mu(\lambda) + \mathcal{O}(L^{-3})\\ &= \frac{1}{L}\mu(\lambda) - \frac{a+b-1}{L^2}\mu(\lambda) + \frac{D}{8L^2}\mathcal{F}\left(\frac{\sigma''}{2D^2}\mu(\lambda)\right) + \mathcal{O}(L^{-3}).\end{aligned} \tag{19}$$

Note that this result is compatible with the exact expressions of the first three cumulants of the current obtained in [35].

The KMP model is also a lattice model in which L harmonic oscillators whose positions x_j are coupled (we use the Itô convention and the Giardinà *et al.* [40] version of the KMP model),

$$2 \le j \le N-1, \quad \frac{dx_j}{dt} = -x_j + x_{j+1}\eta_{j,j+1} - x_{j-1}\eta_{j-1,j}, \tag{20}$$

and the chain is in contact at both ends with heat baths imposing temperatures T_1 and T_L,

$$\frac{dx_1}{dt} = -\left(\gamma_1 + \frac{1}{2}\right)x_1 - \sqrt{2\gamma_1 T_1}\xi_1 + x_2\eta_{1,2},$$

$$\frac{dx_L}{dt} = -\left(\frac{1}{2} + \gamma_L\right)x_L + \sqrt{2\gamma_L T_L}\xi_L - x_{L-1}\eta_{L-1,L}, \tag{21}$$

where ξ_1, ξ_L, and $\eta_{j,j+1}$ (for $1 \le j \le L-1$) are Gaussian white noises with variance unity, and γ_1, γ_L set the time scale of the energy exchange with each reservoir. We refer the reader to Giardinà *et al.* [40,41] for further details and connections between the SSEP and KMP. It is also shown in Appendix B that for the KMP model we have

$$\begin{aligned}\psi(s) &= \frac{1}{L}\mu(\lambda) - \frac{\frac{1}{2\gamma_1}+\frac{1}{2\gamma_L}-1}{L^2}\mu(\lambda)\\ &\quad + \frac{D}{8L^2}\mathcal{F}\left(\frac{\sigma''}{2D^2}\mu(\lambda)\right) + \mathcal{O}(L^{-3}).\end{aligned} \tag{22}$$

Both Eqs. (19) and (22) reveal that taking into account microscopic details of the systems leads, as expected, to nonuniversal corrections to the current large deviation function. Whatever the form of these nonuniversal contributions to $\psi(s)$, it can be seen that the relevant piece of information regarding the possibility of a phase transition is contained in the universal part of $\psi_{\rm FH}(s)$. Universality issues are independent of these microscopic corrections. Note that in periodic boundary conditions, such corrections also exist, but they are fully contained in the second cumulant [28].

D. Is a dynamic phase transition possible?

For systems having a constant diffusion constant D (which we set to $D=1$) and a quadratic $\sigma(\rho)=c_2\rho^2+c_1\rho$, the explicit expression of $\mu(\lambda)$ obtained in Eq. (9) allows us to probe criterion (18) for the existence of a phase transition. Working at fixed $\lambda=sL$, no first-order phase transition can occur because condition (18), or equivalently, $\omega(\lambda,\rho_0,\rho_1)=1$, cannot be fulfilled on the real axis of λ. In the original variable s, however, things are different, since in the large system-size limit, and for $c_2<0$ only, the singularity in the complex plane of s eventually hits the real axis at $s=0$. To be more explicit, using Eq. (10), one notices that for $c_2<0$,

$$\lambda \to \infty, \quad \mu(\lambda) \simeq -\frac{c_1^2}{2c_2}\lambda^2, \tag{23}$$

so that after inserting into Eq. (9) and taking the asymptotics, one arrives at

$$\lim_{L\to\infty}\frac{\psi_Q(s)}{L} = \frac{2c_1^2}{c_2}s^2 + \frac{c_1^3}{3\pi}|s|^3 + o(s^3). \tag{24}$$

The singularity at $s=0$ reflects the existence of a dynamic transition in terms of the total particle current but of higher order. The same transition existed for systems with periodic boundary conditions [see Eq. (62) of [28]] and was noted earlier by Lebowitz and Spohn [in Eq. (A.12) of [15]]. The effects of this transition can be seen on the correlation functions [38] which become long ranged. Note also that in this scaling limit (24) does not depend on the reservoir densities anymore because the optimal profile able to carry such large currents settles to density $\frac{1}{2}$ but in vanishingly small region around the system boundaries.

For systems with $c_2>0$, that is, systems with attractive interactions, such as the KMP model [for which $\sigma(\rho)=4\rho^2$], no phase transition can be observed, but the trivial one occurring at infinite densities (akin to a Bose condensation). There exists a set of numerical simulations by Hurtado and Garrido [37] for the KMP model which actually confirm that no phase transition is observed. This negative result is in contrast to—but does not contradict—that in [9,28], in which it was shown that a phase transition exists, for periodic boundary conditions, when $\sigma''>0$.

IV. OUTLOOK

We have shown that in a family of diffusive systems driven out of equilibrium by a chemical-potential gradient, the total particle current exhibits universal fluctuations. These belong to the Edwards-Wilkinson universality class and they are of the same form as that previously found in closed equilibrium systems. Our results apply to diffusive systems characterized by a constant D and a quadratic σ. We have used a mapping of the system fluctuations to those of an equivalent open system in equilibrium. We have hints that this mapping can be extended beyond quadratic fluctuations; for the SSEP, we can actually prove that a similar mapping applies to the full process [42]. Our main concern lies in that our results are indeed limited to the case D constant and σ quadratic. It would be of great interest to find out whether similar universal properties hold for generic D and σ. Perhaps this is a fortuitous coincidence, but Tailleur *et al.* [39] ran into similar restrictions when mapping the density profile large deviations in boundary-driven diffusive systems onto their equilibrium counterparts. Here we see subjects for future research.

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APPENDIX A

In this appendix we prove Eqs. (9) and (10). We assume that

$$D(\rho)=1, \quad \sigma(\rho)=c_2\rho^2+c_1\rho, \tag{A1}$$

with $c_1>0$ and the boundary conditions $\rho(0)=\rho_0$ and $\rho(1)=\rho_1$. In order to find the explicit expression of $\mu(\lambda)$ we start from the implicit equation found in [24] specialized to $D=1$, namely,

$$\mu(\lambda)=-K\left[\int_{\rho_0}^{\rho_1} d\rho\frac{1}{\sqrt{1+2K\sigma}}\right]^2, \tag{A2}$$

with K determined by

$$\lambda=\int_{\rho_0}^{\rho_1} d\rho\frac{1}{\sigma}\left[\frac{1}{\sqrt{1+2K\sigma}}-1\right]. \tag{A3}$$

We know that the optimal profile verifies

$$\partial_x\rho=q\sqrt{1+2K\sigma}, \tag{A4}$$

the solution of which takes the form

$$\rho(x)=-\frac{c_1}{c_2}+f\sinh\{2[\theta_0+(\theta_1-\theta_0)x]\} \tag{A5}$$

provided that f, θ_0, and θ_1 verify

$$(\theta_1-\theta_0)^2=\frac{1}{2}c_2Kq^2, \tag{A6}$$

$$f^2=\frac{2c_2-c_1^2K}{4c_2^2K}, \tag{A7}$$

and the boundary conditions $\rho(0)=\rho_0$ and $\rho(1)=\rho_1$. One performs the change of variable

$$dx=\frac{1}{q}\frac{d\rho}{\sqrt{1+2K\sigma}} \tag{A8}$$

in Eqs. (A2) and (A3). This yields

$$\mu(\lambda)=-\frac{2}{c_2}(\theta_1-\theta_0)^2, \tag{A9}$$

$$\lambda=+\frac{2}{c_1}\ln\frac{c_1\rho_0\cosh 2\theta_1-\sqrt{c_1^2+4c_2^2f^2}\rho_0\sinh 2\theta_1}{c_1\rho_1\cosh 2\theta_0-\sqrt{c_1^2+4c_2^2f^2}\rho_1\sinh 2\theta_0}, \tag{A10}$$

where we have used Eq. (A7) to eliminate K in favor of f, together with the boundary conditions

$$\rho_0=-\frac{c_1}{2c_2}+f\sinh 2\theta_0, \quad \rho_1=-\frac{c_1}{2c_2}+f\sinh 2\theta_1. \tag{A11}$$

We are left with eliminating f, θ_0, and θ_1 from Eqs. (A9)–(A11). Isolating f^2 first from Eq. (A10) one obtains

$$f^2=\frac{c_1^2}{4c_2^2}\frac{z^2\rho_0^2-2z\rho_0\rho_1\cosh 2(\theta_1-\theta_0)+\rho_1^2}{(z\rho_0\sinh 2\theta_1-\rho_1\sinh 2\theta_0)^2}, \tag{A12}$$

where we have set $z=e^{-c_1\lambda/2}$. Grouping f^2 with the square at the denominator in Eq. (A12), one eliminates f using the boundary conditions (A11). This enables us to isolate $\cosh 2(\theta_1-\theta_0)$ in Eq. (A12) and one gets $\sinh^2(\theta_1-\theta_0)=\omega$ with

$$\omega=\frac{c_2}{c_1^2}(1-z^{-1})[c_1(\rho_1-z\rho_0)-c_2(z-1)\rho_0\rho_1], \tag{A13}$$

and finally

$$\mu(\lambda) = \begin{cases} -\dfrac{2}{c_2}(\operatorname{argsinh}\sqrt{\omega})^2 & \text{for } \omega > 0 \\ +\dfrac{2}{c_2}(\arcsin\sqrt{-\omega})^2 & \text{for } \omega < 0. \end{cases} \tag{A14}$$

In the limit $c_1 \to 0$ which is relevant for KMP, ω becomes

$$\omega = \frac{1}{4}c_2\lambda[2(\rho_0 - \rho_1) - c_2\lambda\rho_0\rho_1]. \tag{A15}$$

APPENDIX B

1. Simple symmetric exclusion process

We consider a SSEP on a one-dimensional lattice with L sites, in which particles are injected to the leftmost site $j=1$ (resp. rightmost site $j=L$) with rate α (resp. δ) and removed with rate γ (resp. β). The master operator governing the evolution of a microscopic configuration of occupation numbers $\{n_j\}_{j=1,\ldots,L}$ can be written in the form

$$\mathbb{W}(s) = \sum_{1\le k\le L-1} [e^s\sigma_k^+\sigma_{k+1}^- + e^{-s}\sigma_k^-\sigma_{k+1}^+ - \hat{n}_k(1-\hat{n}_{k+1}) - \hat{n}_{k+1}(1-\hat{n}_k)] + \alpha[e^{-s}\sigma_1^+ - (1-\hat{n}_1)] + \gamma(e^s\sigma_1^- - \hat{n}_1) + \delta[e^{-s}\sigma_L^+ - (1-\hat{n}_L)] + \beta(e^s\sigma_L^- - \hat{n}_L). \tag{B1}$$

In Eq. (B1) we are using a spin basis; the eigenvalue of the Pauli matrix σ_j^z is 1 if site j is occupied and -1 if it is empty ($\hat{n}_j = \frac{1+\sigma_j^z}{2}$ has the eigenvalue $n_j=0$ or 1). We now remark that

$$\exp\left(s\sum_{j=1}^{L} j\hat{n}_j\right)\mathbb{W}(\lambda)\exp\left(-s\sum_{j=1}^{L} j\hat{n}_j\right) = \mathbb{W}_L[s(L+1)], \tag{B2}$$

where $\mathbb{W}_L$ is the operator counting the total current across site L only, the expression of which reads

$$\mathbb{W}_L(s') = \sum_{1\le k\le L-1} [\sigma_k^+\sigma_{k+1}^- + \sigma_k^-\sigma_{k+1}^+ - \hat{n}_k(1-\hat{n}_{k+1}) - \hat{n}_{k+1}(1-\hat{n}_k)] + \alpha[\sigma_1^+ - (1-\hat{n}_1)] + \gamma(\sigma_1^- - \hat{n}_1) + \delta[e^{s'}\sigma_L^+ - (1-\hat{n}_L)] + \beta(e^{-s'}\sigma_L^- - \hat{n}_L), \tag{B3}$$

with s' being conjugate to the time-integrated current through site $i=L$. Owing to Eq. (B2), the largest eigenvalue of $\mathbb{W}_L[s(L+1)]$ is $\psi(s)$, which is the largest eigenvalue of Eq. (B1). We use, for each lattice site, a Holstein-Primakoff-like representation [39],

$$\sigma^+ = 1 - F + F^+ - 2FF^+ + F^2F^+, \quad \sigma^- = F - F^2F^+, \tag{B4}$$

which also leads to $\hat{n} = F + FF^+ - F^2F^+$. The bulk contribution to the evolution operator (B3) now reads

$$\mathbb{W}_{L,\text{bulk}}(\lambda) = -\sum_j [(F_{j+1}-F_j)(F_{j+1}^+ - F_j^+) + (F_{j+1}-F_j)^2F_j^+F_{j+1}^+]. \tag{B5}$$

We represent $e^{\mathbb{W}_L(s')t}$ by means of a path integral [39,43] involving coherent states related to operators F and F^+, which we shall denote by $\phi(\tau)$ and $\bar{\phi}(\tau)$. This leads to an action

$$S_{L,\text{bulk}}[\bar{\phi},\phi] = \int_0^t dt\left\{\sum_{j=1}^{L}\bar{\phi}_j\partial_t\phi_j + \sum_{j=1}^{L-1}[(\phi_{j+1}-\phi_j)(\bar{\phi}_{j+1}-\bar{\phi}_j) + (\phi_{j+1}-\phi_j)^2\bar{\phi}_j\bar{\phi}_{j+1}]\right\}, \tag{B6}$$

while the boundary terms are given by

$$S_{L,\text{boundary}}[\bar{\phi},\phi] = -\int_0^t dt[\alpha\bar{\phi}_1 - (\alpha+\gamma)\bar{\phi}_1\phi_1] + e^{s'}\int_0^t dt([(e^{-s'}\beta+\delta)\phi_L - \delta] \times\{-e^{-s'} + [(e^{-s'}-1)\phi_L + 1]\bar{\phi}_L + 1\}). \tag{B7}$$

Since we are interested in the large-time behavior, we shall proceed with a saddle-point approximation at fixed t but as $L\to\infty$, keeping the system size L fixed (this is possible due to our saddle-point equations being stationary). We use the notation $\nabla_j\phi = \phi_{j+1} - \phi_j$. The saddle-point equation obtained by differentiating S_L with respect to $\bar{\phi}_j$ reads

$$(\nabla_j\bar{\phi} + 2\nabla_j\phi\bar{\phi}_j\bar{\phi}_{j+1}) - (\nabla_{j-1}\bar{\phi} + 2\nabla_{j-1}\phi\bar{\phi}_{j-1}\bar{\phi}_j) = 0, \tag{B8}$$

and thus there exists K_1 such that

$$\nabla_j\phi = \frac{-\nabla_j\bar{\phi} + 2K_1}{2\bar{\phi}_j}. \tag{B9}$$

Writing the variational equation with respect to ϕ_j and using Eq. (B9), we obtain

$$\frac{\bar{\phi}_{j+1}+\bar{\phi}_{j-1}}{\bar{\phi}_{j+1}\bar{\phi}_j^2\bar{\phi}_{j-1}}(4K_1^2 - \bar{\phi}_j^2 + \bar{\phi}_{j+1}\bar{\phi}_{j-1}) = 0, \tag{B10}$$

which we multiply by

$$\frac{\bar{\phi}_{j+1}-\bar{\phi}_{j-1}}{\bar{\phi}_{j+1}+\bar{\phi}_{j-1}}\bar{\phi}_j, \tag{B11}$$

so that

$$\left[\frac{4K_1^2 - (\nabla_j\bar{\phi})^2}{\bar{\phi}_{j+1}\bar{\phi}_j}\right] - \left[\frac{4K_1^2 - (\nabla_{j-1}\bar{\phi})^2}{\bar{\phi}_j\bar{\phi}_{j-1}}\right] = 0, \tag{B12}$$

which leads to the existence of another constant K_2 such that

$$(\nabla_j \bar{\phi}_j)^2 = 4K_1^2 + K_2 \bar{\phi}_j \bar{\phi}_{j+1}. \tag{B13}$$

We thus obtain that when evaluated at the saddle, $S[\bar{\phi},\phi]=-t\frac{L-1}{2}K_2$. Besides, it is possible to solve the bulk saddle-point equations [Eqs. (B9) and (B13)],

$$2 \le j \le L-1, \quad \bar{\phi}_j = -A \sinh[(j-1)B + C],$$
$$\phi_j = E + \frac{1}{2A}\tanh\frac{(j-1)B + C}{2}, \tag{B14}$$

where A, B, and C are related to K_1 and K_2 by $K_1 = -\frac{A}{2}\sinh B$ and $K_2 = 4\sinh^2\frac{B}{2}$. At this stage we write the saddle-point equations corresponding to the fields located at the boundaries $j=1$ and $j=L$. At $j=1$ this leads to

$$0 = \bar{\phi}_2 - \bar{\phi}_1 + 2(\phi_2 - \phi_1)\bar{\phi}_2\bar{\phi}_1 - \alpha\bar{\phi}_1 - \gamma\bar{\phi}_1 \tag{B15}$$

and

$$0 = \phi_2 - \phi_1 - (\phi_2 - \phi_1)^2\bar{\phi}_2 + \alpha(1-\phi_1) - \gamma\phi_1. \tag{B16}$$

This immediately sets the constant E appearing in Eq. (B14) to $E=\frac{\alpha}{\alpha+\gamma}=\rho_0$ and further imposes that $a=\frac{1}{\alpha+\gamma}=\frac{\sinh C}{\sinh B}$. Due to the latter relation between B and C, only two unknowns A and B remain to be determined. This is done by writing the two saddle-point equations at $j=L$ and by substituting solution (B14). The additional constraints on A and B (or C) are

$$A^2 = \frac{(z-1)[z(\rho_1 - 1) - \rho_1]}{A[(z-1)\rho_0 + 1][z\rho_0(\rho_1 - 1) - \rho_0\rho_1 + \rho_1]},$$
$$z = e^{-s'}, \tag{B17}$$

and

$$\sinh[(L-1)B + C + \varepsilon] + \frac{b}{a}\sinh C = 0, \tag{B18}$$

where $\sinh^2\frac{\varepsilon}{2}=\omega$, $\rho_1=\frac{\delta}{\beta+\delta}$ is also the density at site L, and where variable ω is exactly that defined in Eq. (11) with s' instead of λ. Finally, we eliminate C to obtain B as the solution to

$$\sinh^2[(L-1)B + \varepsilon]$$
$$= \{a^2 + b^2 + 2ab\cosh[(L-1)B + \varepsilon]\}\sinh^2 B. \tag{B19}$$

Equation (B19) can be solved in powers of $1/L$; to leading order, B and C are $\mathcal{O}(1/L)$, while ε is $\mathcal{O}(1)$, and thus one has $B=\frac{1}{L}\varepsilon=\frac{2}{L}\text{arcsinh}\sqrt{\omega}$. To the next order one has

$$\psi_L(s') = \frac{1}{2a}(-1 + \sqrt{1 + a^2\sinh^2 B})$$
$$+ \frac{1}{2b}(-1 + \sqrt{1 + b^2\sinh^2 B}) + (L-1)\sinh^2\frac{B}{2}$$
$$\simeq \frac{\mu(s')}{L} - \frac{a+b-1}{L^2}\mu(s') + \mathcal{O}(L^{-3}). \tag{B20}$$

This proves the result announced in Eq. (19).

2. Kipnis-Marchioro-Presutti model

For the KMP process, one writes a Langevin equation for $\varepsilon_i=\frac{1}{2}x_i^2$ based on Eq. (20). Using the Itô discretization scheme, this leads to

$$\frac{d\varepsilon_i}{dt} = j_i - j_{i+1}, \tag{B21}$$

where the local energy current is $j_{i+1}=\varepsilon_i-\varepsilon_{i+1}+2\sqrt{\varepsilon_i\varepsilon_{i+1}}\,\eta_{i,i+1}(1\le i\le L-2)$, and $j_1=\gamma_1T_1-2\gamma_1\varepsilon_1+2\sqrt{\gamma_1T_1}\xi_1$, $j_{L+1}=-\gamma_LT_L+2\gamma_L\varepsilon_L+2\sqrt{\gamma_LT_L}\xi_L$. Using the Janssen–De Dominicis formalism, one is again led to

$$\langle e^{-sQ}\rangle = \int \mathcal{D}\bar{\varepsilon}_j\mathcal{D}\varepsilon_j e^{-S[\bar{\varepsilon}_j,\varepsilon_j]}, \tag{B22}$$

where the action has the expression

$$S = \int dt\sum_{j=1}^{L}\bar{\varepsilon}_j\partial_t\varepsilon_j + \int dt\sum_{j=1}^{L-1}[(\bar{\varepsilon}_{j+1} - \bar{\varepsilon}_j - s)(\varepsilon_{j+1} - \varepsilon_j)$$
$$- 2\varepsilon_j\varepsilon_{j+1}(\bar{\varepsilon}_{j+1} - \bar{\varepsilon}_j - s)^2]$$
$$+ 2\gamma_1\int dt\{-T_1(\bar{\varepsilon}_1 - s)[(\bar{\varepsilon}_1 - s)\varepsilon_1 + 1/2] + (\bar{\varepsilon}_1 - s)\varepsilon_1\}$$
$$+ 2\gamma_L\int dt\{-T_L(\bar{\varepsilon}_L + s)[(\bar{\varepsilon}_L + s)\varepsilon_L + 1/2] + (\bar{\varepsilon}_L + s)\varepsilon_L\}. \tag{B23}$$

With the change $\bar{\varepsilon}_j'=\bar{\varepsilon}_j-sj$, and dropping the primes, the action becomes

$$S = \int dt\sum_{j=1}^{L}\bar{\varepsilon}_j\partial_t\varepsilon_j + \int dt\sum_{j=1}^{L-1}[(\bar{\varepsilon}_{j+1} - \bar{\varepsilon}_j)(\varepsilon_{j+1} - \varepsilon_j)$$
$$- 2\varepsilon_j\varepsilon_{j+1}(\bar{\varepsilon}_{j+1} - \bar{\varepsilon}_j)^2]$$
$$+ 2\gamma_1\int dt[-T_1\bar{\varepsilon}_1(\bar{\varepsilon}_1\varepsilon_1 + 1/2) + \bar{\varepsilon}_1\varepsilon_1]$$
$$+ 2\gamma_L\int dt(-T_L[\bar{\varepsilon}_L + s(L+1)]\{[\bar{\varepsilon}_L + s(L+1)]\varepsilon_L + 1/2\}$$
$$+ [\bar{\varepsilon}_L + s(L+1)]\varepsilon_L), \tag{B24}$$

which shows that $\langle e^{-sQ}\rangle=\langle e^{-s(L+1)Q_L}\rangle$, where Q_L is the time-integrated current flowing between site L and the right thermal bath. We shall denote $s'=(L+1)s$. An additional change of fields, which leaves the bulk part invariant (see [39]), allows one to further simplify the boundary terms; we set $\varepsilon'=\frac{2\varepsilon}{1+2\bar{\varepsilon}\varepsilon}$ and $\bar{\varepsilon}'=\frac{1}{2}\bar{\varepsilon}(1+2\bar{\varepsilon}\varepsilon)$, and we obtain (dropping the primes)

$$S=\int dt\sum_{j=1}^{L}\bar{\varepsilon}_j\partial_t\varepsilon_j+\int dt\sum_{j=1}^{L-1}[(\bar{\varepsilon}_{j+1}-\bar{\varepsilon}_j)(\varepsilon_{j+1}-\varepsilon_j)-2\varepsilon_j\varepsilon_{j+1}(\bar{\varepsilon}_{j+1}-\bar{\varepsilon}_j)^2]+2\gamma_1\int dt[-T_1\varepsilon_1+\bar{\varepsilon}_1\varepsilon_1]-\gamma_L\int dt\{[s'+2\varepsilon_L(1+s'\bar{\varepsilon}_L)][T_L+\bar{\varepsilon}_L(s'T_L-1)]\}. \quad (B25)$$

We differentiate S given in Eq. (B25) with respect to $\bar{\varepsilon}_j$,

$$[\nabla_j\varepsilon-4\nabla_j\bar{\varepsilon}\varepsilon_j\varepsilon_{j+1}]-[\nabla_{j-1}\varepsilon-4\nabla_{j-1}\bar{\varepsilon}\varepsilon_{j-1}\varepsilon_j]=0, \quad (B26)$$

where we used the notation $\nabla_jX=X_{j+1}-X_j$. One thus has a constant K_1 such that

$$\nabla_j\bar{\varepsilon}=\frac{K_1+\nabla_j\varepsilon}{4\varepsilon_j\varepsilon_{j+1}}. \quad (B27)$$

Differentiating now Eq. (B25) with respect to ε_j one has

$$\nabla_j\bar{\varepsilon}-\nabla_{j-1}\bar{\varepsilon}+2(\nabla_j\bar{\varepsilon})^2\varepsilon_{j+1}+2(\nabla_{j-1}\bar{\varepsilon})^2\varepsilon_{j-1}=0, \quad (B28)$$

and substituting Eq. (B27) to get an equation on the ε_j's only, one obtains

$$\frac{\varepsilon_{j+1}+\varepsilon_{j-1}}{\varepsilon_{j+1}\varepsilon_j^2\varepsilon_{j-1}}(K_1^2-\varepsilon_j^2+\varepsilon_{j+1}\varepsilon_{j-1})=0. \quad (B29)$$

The trick is to multiply Eq. (B29) by

$$\frac{\varepsilon_{j+1}-\varepsilon_{j-1}}{\varepsilon_{j+1}+\varepsilon_{j-1}}\varepsilon_j, \quad (B30)$$

which leads to

$$\left[\frac{K_1^2-(\nabla_j\varepsilon)^2}{\varepsilon_{j+1}\varepsilon_j}\right]-\left[\frac{K_1^2-(\nabla_{j-1}\varepsilon)^2}{\varepsilon_j\varepsilon_{j-1}}\right]=0, \quad (B31)$$

and thus there exists a constant K_2 such that

$$(\nabla_j\varepsilon)^2=K_1^2+4K_2\varepsilon_j\varepsilon_{j+1}. \quad (B32)$$

We substitute Eq. (B27) into the bulk part of action (B25), and we arrive at

$$-\frac{1}{t}S_{\text{bulk}}=\sum_{1\le j\le L-1}\frac{K_1^2-(\nabla_j\varepsilon)^2}{8\varepsilon_{j+1}\varepsilon_j}=-\frac{L-1}{2}K_2. \quad (B33)$$

We differentiate the action with respect to the fields at boundaries $\bar{\varepsilon}_1$ and ε_1,

$$-(\varepsilon_2-\varepsilon_1)+4(\bar{\varepsilon}_2-\bar{\varepsilon}_1)\varepsilon_1\varepsilon_2+2\gamma_1\varepsilon_1=0, \quad (B34)$$

$$-(\bar{\varepsilon}_2-\bar{\varepsilon}_1)-2(\bar{\varepsilon}_2-\bar{\varepsilon}_1)^2\varepsilon_2+2\gamma_1(\bar{\varepsilon}_1-T_1)=0. \quad (B35)$$

Differentiating with respect to $\bar{\varepsilon}_L$ and ε_L one gets

$$(\varepsilon_L-\varepsilon_{L-1})-4(\bar{\varepsilon}_L-\bar{\varepsilon}_{L-1})\varepsilon_L\varepsilon_{L-1}+\gamma_L[2\varepsilon_L+(1-4T_L\varepsilon_L+4\varepsilon_L\bar{\varepsilon}_L)\lambda-T_L(1+4\varepsilon_L\bar{\varepsilon}_L)s'^2]=0, \quad (B36)$$

$$(\bar{\varepsilon}_L-\bar{\varepsilon}_{L-1})-2(\bar{\varepsilon}_L-\bar{\varepsilon}_{L-1})^2\varepsilon_{L-1}-2\gamma_L(1+s'\bar{\varepsilon}_L)[T_L+\bar{\varepsilon}_L(\lambda T_L-1)]=0. \quad (B37)$$

We now proceed with solving the microscopic equations [Eqs. (B27) and (B32)]. We search for a solution in the form

$$\varepsilon_j=A\sinh\{2[(j-1)B+C]\},$$

$$\bar{\varepsilon}_j=E+\frac{1}{4A}\tanh[(j-1)B+C], \quad (B38)$$

where A, B, C, and E are four constants to be determined by the four saddle-point equations at the boundaries. We first note that, quite remarkably, Eq. (B38) is an exact solution of the microscopic bulk saddle-point equations [Eqs. (B27) and (B32)], on the condition that

$$K_1=-A\sinh(2B),\quad K_2=\sinh^2B. \quad (B39)$$

One checks that the saddle equations at site 1 are solved by

$$E=T_1,\quad \frac{1}{2}\frac{\sinh 2B}{\sinh 2C}=\gamma_1. \quad (B40)$$

Eliminating γ_L between the saddle equations at site L yields

$$A^2=\frac{s'(T_Ls'-1)}{16(T_1s'+1)[T_L+T_1(T_Ls'-1)]}. \quad (B41)$$

Substituting this result into Eq. (B36), one gets

$$\sinh\{2[(L-1)B+C+\varepsilon]\}+\frac{\gamma_1}{\gamma_L}\sinh 2C=0, \quad (B42)$$

where ε is such that

$$\sinh^2\varepsilon=\omega, \quad (B43)$$

and ω is given by

$$\omega=\lambda(T_1-T_L-\lambda T_1T_L), \quad (B44)$$

in accordance with Eq. (12). One can now eliminate C using Eq. (B40) and this gives an equation involving B only

$$\sinh^2[2(L-1)B+2\varepsilon]=\frac{\gamma_1^{-2}+\gamma_L^{-2}+2(\gamma_1\gamma_L)^{-1}\cosh[2(L-1)B+2\varepsilon]}{4}\sinh^2 2B. \quad (B45)$$

The large deviation function is given by the value of $\mathbb{W}_{Q_L}$ at saddle. Combining the bulk contribution (B33) together with the boundary terms read from Eq. (B25), one obtains

$$\psi_{Q_L}(\lambda)=\frac{1}{2}\gamma_1-\frac{1}{2}\sqrt{\gamma_1^2+\sinh^2 2B}+\frac{1}{2}\gamma_L-\frac{1}{2}\sqrt{\gamma_L^2+\sinh^2 2B}-\frac{L-1}{2}\sinh^2B, \quad (B46)$$

where B is solution of Eq. (B45).

Though the expressions are cumbersome, one can still solve Eq. (B45) perturbatively in powers of L by writing B in the form $B=B_0/L+B_1/L^2+\cdots$. To lowest order, B is of order

$1/L$ and ε of order 1; in Eq. (B45), the right-hand term is negligible and one obtains $B_0=-\varepsilon=-\text{arcsinh}\sqrt{\omega}$, which yields the macroscopic fluctuation theory result $\mu(\lambda) = -\frac{1}{2}(\text{arcsinh}\sqrt{\omega})^2$ found previously in Eqs. (9) and (12), as expected. To the next order, one gets

$$\psi_{Q_L}(\lambda) = \frac{1}{L}\mu(\lambda) + \frac{1}{L^2}\left[\left(1 - \frac{1}{2\gamma_1} - \frac{1}{2\gamma_L}\right)\mu(\lambda)\right] + \mathcal{O}(L^{-3}), \quad \text{(B47)}$$

which matches the announced result (22).

[1] C. Kipnis, S. Olla, and S. Varadhan, Commun. Pure Appl. Math. **42**, 115 (1989).
[2] H. Spohn, *Large Scale Dynamics of Interacting Particles* (Springer-Verlag, Berlin, 1991).
[3] T. Liggett, *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes*, Fundamental Principles of Mathematical Sciences Vol. 324 (Springer-Verlag, Berlin, 1999).
[4] C. Kipnis and C. Landim, *Scaling Limits of Interacting Particle Systems* (Springer, New York, 1999).
[5] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, Phys. Rev. Lett. **87**, 040601 (2001).
[6] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, J. Stat. Phys. **107**, 635 (2002).
[7] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, Math. Phys., Anal. Geom. **6**, 231 (2003).
[8] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, Phys. Rev. Lett. **94**, 030601 (2005).
[9] L. Bertini, A. De Sole, D. Gabrielli, G. Jona-Lasinio, and C. Landim, J. Stat. Phys. **123**, 237 (2006).
[10] A. Einstein, Ann. Phys. **17**, 549 (1905).
[11] R. Kubo, M. Toda, and N. Hashitsume, *Statistical Physics II. Nonequilibrium Statistical Mechanics*, Springer Series in Solid-State Sciences Vol. 31 (Springer, New York, 1998).
[12] D. J. Evans, E. G. D. Cohen, and G. P. Morriss, Phys. Rev. Lett. **71**, 2401 (1993).
[13] G. Gallavotti and E. G. D. Cohen, Phys. Rev. Lett. **74**, 2694 (1995).
[14] J. Kurchan, J. Phys. A **31**, 3719 (1998).
[15] J. L. Lebowitz and H. Spohn, J. Stat. Phys. **95**, 333 (1999).
[16] C. Maes, F. Redig, and A. V. Moffaert, J. Math. Phys. **41**, 1528 (2000).
[17] U. Seifert, Phys. Rev. Lett. **95**, 040602 (2005).
[18] C. Maes, J. Stat. Phys. **95**, 367 (1999).
[19] G. Gallavotti, Chaos **14**, 680 (2004).
[20] P. Gaspard, J. Chem. Phys. **120**, 8898 (2004).
[21] B. Derrida, J. L. Lebowitz, and E. R. Speer, Phys. Rev. Lett. **87**, 150601 (2001).
[22] B. Derrida, J. L. Lebowitz, and E. R. Speer, J. Stat. Phys. **107**, 599 (2002).
[23] B. Derrida, J. L. Lebowitz, and E. R. Speer, Phys. Rev. Lett. **89**, 030601 (2002).
[24] T. Bodineau and B. Derrida, Phys. Rev. Lett. **92**, 180601 (2004).
[25] T. Bodineau and B. Derrida, Phys. Rev. E **72**, 066110 (2005).
[26] T. Bodineau and B. Derrida, C. R. Phys. **8**, 540 (2007).
[27] B. Derrida and J. L. Lebowitz, Phys. Rev. Lett. **80**, 209 (1998).
[28] C. Appert-Rolland, B. Derrida, V. Lecomte, and F. van Wijland, Phys. Rev. E **78**, 021122 (2008).
[29] C. Kipnis, C. Marchioro, and E. Presutti, J. Stat. Phys. **27**, 65 (1982).
[30] L. Bertini, D. Gabrielli, and J. L. Lebowitz, J. Stat. Phys. **121**, 843 (2005).
[31] S. F. Edwards and D. R. Wilkinson, Proc. R. Soc. London, Ser. A **381**, 17 (1982).
[32] H. K. Janssen, in *From Phase Transitions to Chaos*, edited by G. Györgyi, I. Kondor, L. Sasvári, and T. Tél (World Scientific, Singapore, 1992).
[33] J. Kurchan, *Six Out-of-Equilibrium Lectures* (Oxford University Press, New York, 2008), Sec. 4.1.
[34] J. Tailleur, J. Kurchan, and V. Lecomte, J. Phys. A **41**, 505001 (2008).
[35] B. Derrida, B. Douçot, and P.-E. Roche, J. Stat. Phys. **115**, 717 (2004).
[36] A. N. Jordan, E. V. Sukhorukov, and S. Pilgram, J. Math. Phys. **45**, 4386 (2004).
[37] P. I. Hurtado and P. L. Garrido, Phys. Rev. Lett. **102**, 250601 (2009).
[38] T. Bodineau, B. Derrida, V. Lecomte, and F. van Wijland, J. Stat. Phys. **133**, 1013 (2008).
[39] J. Tailleur, J. Kurchan, and V. Lecomte, Phys. Rev. Lett. **99**, 150602 (2007).
[40] C. Giardinà, J. Kurchan, and F. Redig, J. Math. Phys. **48**, 033301 (2007).
[41] C. Giardinà, J. Kurchan, F. Redig, and K. Vafayi,J. Stat. Phys. **135**, 25 (2009).
[42] V. Lecomte *et al.* (unpublished).
[43] M. Doi, J. Phys. A **9**, 1465 (1976).